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SPACE CHARGE LIMITATION ON THE FOCUS OF ELECTRON BEAMS

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Several papers discussing the effect of mutual electron repulsion on the focus of electron beams have appeared in the past. One of the better known of these is that of Thompson and Headrick.[1]

In cathode ray tubes or other similar electron devices, the smallest spot at the screen is not necessarily obtained by focusing the beam in the plane of the screen. If beam space charge is a significant factor, the smallest spot is obtained by focusing on a plane between the gun and screen. Thompson and Headrick have discussed this phenomenon and indicate how the minimum spot size may be obtained. Their procedure involves the plotting of a multiplicity of beam envelopes for each particular set of electrical and geometrical conditions.

An analytical method of finding the minimum spot size is presented here.[2] Furthermore, a single universal expression, one that is valid for any current, voltage, beam length, etc. is obtained.

Statement of the Problem

Fig. 1 shows the superposition of two idealized electron beams.[3] The beams have equal currents, voltages, initial diameters and axial lengths. One beam, however, is focussed in such a manner that its minimum cross-section occurs at the screen. The other beam has its minimum cross-section somewhat ahead of the screen. Since the focus point of the first occurs at a larger axial distance from the gun than that of the second, the latter is of necessity the smaller of the two. In fact, the cross-section at the screen of the second beam is smaller than that of the beam whose minimum cross-section occurs at the screen.

The purpose of this paper is to present a derivation of an expression for the beam size at the screen when the beam is focussed in such a manner as to cause the smallest possible spot at the screen.

Many readers may not care to follow the rather lengthy mathematical details. For this reason an outline of the derivation and the final expression are presented in the following section and the complete derivation deferred to the appendix.

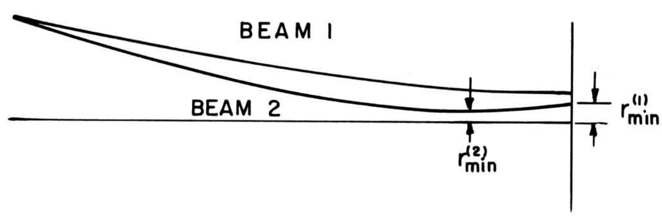

Fig. 1 – Cross section of electron beams. Beam 1 is focused for minimum beam cross section at the screen; beam 2 for smallest spot size at the screen.

Universal Expression for the Smallest Spot Size

The time of flight, T, of an electron path coincident with the outside contour of an electron beam is given by the following expression:

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1. Thompson, B.J. and L.B. Headrick, "Space Charge Limitations on the Focus of Electron Beams," *Proc. I.R.E.*, Vol. 28, July 1940, pp. 318-324.
2. Since the completion of this work, it has come to the author's attention that a somewhat equivalent treatment has been developed by D. L. Hollway: "The Optimum Space-Charge Controlled Focus of an Electron Beam" – *Australian Journal of Scientific Research*, Series A, Vol. 5, 1952, pp. 430-436.
3. The electron beams discussed in this paper are idealized in certain respects. Thermal velocities, finite source size, etc. are neglected. See reference 1 for a complete statement of the customary assumptions.

$$T = \pm \frac{1}{K} \int_{\substack{\text{Beam} \\ \text{Contour}}} \frac{dr}{\sqrt{\log \frac{r}{r_i} + \frac{\dot{r}_i^2}{K^2}}} \qquad (10)$$

Here K is a constant which depends on beam current and voltage, r the instantaneous radial displacement of the electron along the path and r_i and $\dot{r}_i$ the initial beam radius and slope respectively. For a fixed beam length and voltage, T is determined. The path of the integral depends upon beam voltage, current, and upon $\dot{r}_i$ and r_i. The final limit of the integral is r_s, the spot size on the screen. Eq. (10) is differentiated with respect to $\dot{r}_i$ holding the other parameters constant in order to find the path yielding the stationary value of r_s. A transcendental equation involving only the normalized spot size r_s/r_i and the quantity $\dot{r}_i/K$ is obtained.

Fig. 2 is a graph[1] of a numerical solution to the transcendental equation. It is a universal relationship for the minimum spot size obtainable at the screen for any choice of beam length, Z; voltage, V; current, I (amperes); and initial diameter, $2r_i$. A curve which indicates the spot size for the condition where the beam is focused in such a manner that its minimum cross section occurs in the plane of the screen is also shown in Fig. 2. At low currents the two curves coincide. At larger currents, however, the smallest spot size at the screen is appreciably smaller than the spot obtained when the minimum beam cross section is made to occur at the screen. In fact, beyond a limiting current of about $4 \times 10^{-5} V^{3/2} r_i^2/Z^2$ (amperes), the minimum beam cross section cannot be made to occur in the plane of the screen.

The abscissa of Fig. 2 is the ratio of the final and initial beam radii and consequently is dimensionless.

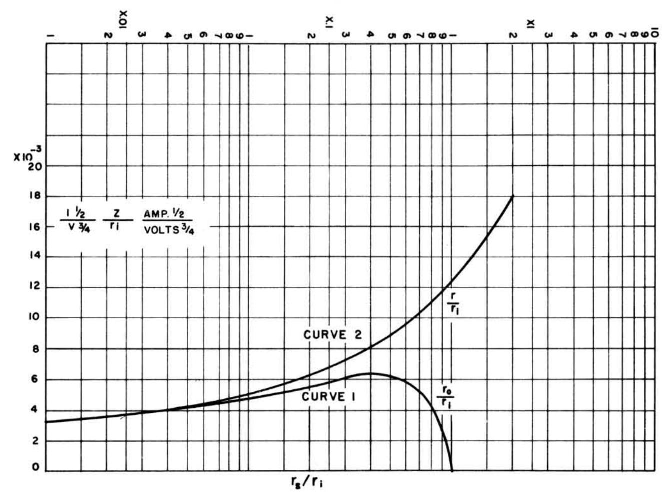
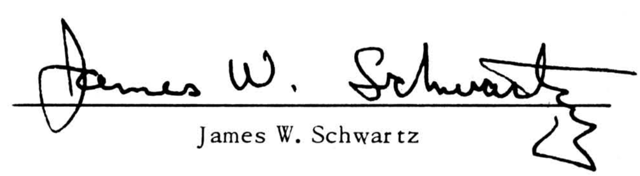


Fig. 2 – Universal relationships for beams with minimum beam diameter at the screen (curve 1) and for minimum spot size at the screen (curve 2).

The ordinate, however, is proportional to the square root of the current and inversely proportional to the 3/4 power of the beam potential. Since this is not dimensionless, the curve applies only when the current is expressed in amperes and the potential in volts. For example, at a current of 10^{-2} amperes, a potential of 10^4 volts, beam length of 10 inches and initial radius of 8×10^{-2} inches the ordinate is $\frac{I^{1/2}}{V^{3/4}} \times \frac{Z}{r_i} = \frac{10^{-1}}{10^{+3}} \times \frac{10}{8 \times 10^{-2}} = 12.5 \times 10^{-3}$.

Referring to the curves of Fig. 2, it is seen that the smallest obtainable spot size is 1.0 times the initial beam size. Curve No. 1 indicates that the beam current is about four times too high to allow the smallest beam cross section to occur at the screen.



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[1] An enlarged copy of this graph appears on Page 8.

Derivation of Equations

The following symbols appear in the derivation:

Symbol	*Meaning*
r	Instantaneous radius of an electron on the edge of the beam.
$\dot{r}$	First time derivative of r.
$\ddot{r}$	Second time derivative of r.
r_o	Beam radius at the beam minimum cross-section.
r_i	Initial beam radius, radius at the gun.
r_s	Beam radius at the screen.
Z	Gun to screen axial distance.
t	Time (seconds)
T	Gun to screen electron transit time.
I	Beam current (amperes)
V	Beam potential (volts)
η	Electronic charge to mass ratio ($\eta = e/m = 1.76 \times 10^{11}$ *coulumb/KG*)
ϵ	Permitivity of vacuum ($\epsilon = 8.85 \times 10^{-12}$ *farads/meter*)
K	$K \equiv \sqrt{\eta I/\pi\epsilon\sqrt{2\eta V}}$ *meters/second*
u	An integration variable $u \equiv \pm\sqrt{\log r/r_i + \dot{r}_i^2/K^2}$
u_s	$u_s \equiv \sqrt{\log r_s/r_i + \dot{r}_i^2/K^2}$
u_o	$u_o = \sqrt{\log r_o/r_i + \dot{r}_i/K^2} = 0$
$U(x)$	A non-elementary function, $U(x) \equiv \int_0^x e^{u^2} du$
W	$W \equiv \dot{r}_i/K$
p	$p \equiv r_s/r_i$
$f(W,p)$	$f(W,p) \equiv 2\,We^{-W^2}[U(u_s) + U(W)]$
$g(W,p)$	$g(W,p) \equiv 1 + p\,W/u_s$
β	Value of W for which $g(W,p) = f(W,p)$

Fig. 3 illustrates the usage of the geometrical symbols.

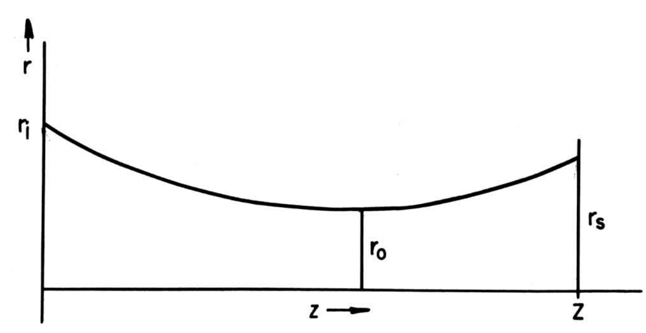

Fig. 3 – Beam cross section showing usage of geometrical symbols.

By a simple application of Gauss's law the electric field just outside a long cylinder of charge is found to be

$$\frac{I}{2\pi\epsilon r\sqrt{2\eta V}}$$

Hence, in an otherwise field-free region the motion of electrons under the influence of space charge must satisfy the following relation:

$$\ddot{r} - \frac{\eta I}{2\pi\epsilon r\sqrt{2\eta V}} = 0 \tag{1}$$

If Eq. (1) be multiplied by $\dot{r}$, there results

$$\ddot{r}\dot{r} - \frac{\eta I}{2\pi\epsilon\sqrt{2\eta V}}\frac{\dot{r}}{r} = 0. \tag{2}$$

Or, for simplicity

$$\ddot{r}\dot{r} - \frac{K^2}{2}\frac{\dot{r}}{r} = 0 \tag{3}$$

But, Eq. (3) is an exact differential

$$1/2\,\frac{d}{dt}\left\{(\dot{r})^2 - K^2 \log r\right\} = 0. \tag{4}$$

Hence,

$$(\dot{r})^2 - K^2 \log r = const. \tag{5}$$

Substituting the initial conditions

$$\left.\begin{array}{l} r = r_i \\ \dot{r} = \dot{r}_i \\ t = 0 \end{array}\right\} \tag{6}$$

$$\dot{r}^2 - \dot{r}_i^2 - K^2 \log\frac{r}{r_i} = 0. \tag{7}$$

This may be written as

$$\frac{dr}{dt} = \sqrt{K^2 \log\frac{r}{r_i} + \dot{r}_i^2}, \tag{8}$$

$$dt = \frac{dr}{\sqrt{K^2 \log\frac{r}{r_i} + \dot{r}_i^2}} \tag{9}$$

Integrating

$$KT = \pm\int_{Beam} \frac{dr}{\sqrt{\log\frac{r}{r_i} + \frac{\dot{r}_i^2}{K^2}}} \tag{10}$$

Since r first decreases from r_i to r_o and then increases to some value r_s, the integral must read

$$KT = -\int_{r_i}^{r_o} \frac{dr}{\sqrt{\log \frac{r}{r_i} + \frac{\dot{r}_i^2}{K^2}}} + \int_{r_o}^{r_s} \frac{dr}{\sqrt{\log \frac{r}{r_i} + \frac{\dot{r}_i^2}{K^2}}} \tag{11}$$

For convenience let

$$u^2 \equiv \log \frac{r}{r_i} + \frac{\dot{r}_i^2}{K^2}, \tag{12}$$

then

$$2u\, du = \frac{dr}{r}, \tag{13}$$

$$r = r_i\, e^{u^2 - \dot{r}_i^2/K^2}. \tag{14}$$

Also,

$$u = \pm \sqrt{\log \frac{r}{r_i} + \frac{\dot{r}_i^2}{K^2}}\ . \tag{15}$$

The sign of the square root was taken as negative in the first term and positive in the second term of Eq. (11). This was necessary since both sides of Eq. (11) must increase monotonically as an electron traces out a path in time, t. For consistency then, the same sign selection must be made in the application of Eq. (15) to the two regions.

Substituting Eqs. (13), (14) and (15) into Eq. (11) gives

$$KT = 2r_i\, e^{-\dot{r}_i^2/K^2} \left[\int_{u_o}^{-\dot{r}_i/K} e^{u^2}\, du + \int_{u_o}^{u_s} e^{u^2}\, du \right] \tag{16}$$

When $r = r_o$, however, $dr/dt = 0$. Hence setting Eq. (8) equal to zero one obtains

$$K^2 \log \frac{r_o}{r_i} = -\dot{r}_i^2\ , \tag{17}$$

or

$$r_o = r_i\, e^{-\dot{r}_i^2/K^2}\ . \tag{18}$$

Defining $-\dot{r}_i/K \equiv W$, a positive number, and using Eq. (18), Eq. (16) simplifies to

$$KT = 2r_i\, e^{-W^2} \left[\int_0^W e^{u^2}\, du + \int_0^{u_s} e^{u^2}\, du \right] \tag{19}$$

Define $\sqrt{\log r_s/r_i + W^2}$ as u_s, a positive number.

Also, let $U(x)$ be defined by the equation

$$U(x) \equiv \int_0^x e^{u^2}\, du \tag{20}$$

Then Eq. (19) becomes

$$KT = 2r_i\, e^{-W^2} \left[U(W) + U(u_s) \right]. \tag{21}$$

Remembering that u_s is a function of W, Eq. (21) is differentiated with respect to W. This is equivalent to differentiation with respect to $\dot{r}_i$ for fixed beam voltage, current, beam length and initial diameter. The result is set equal to zero in order to obtain the stationary condition with respect to $\dot{r}_i$.

$$-4Wr_i\, e^{-W^2} \left[U(W) + U(u_s) \right] + 2r_i\, e^{-W^2} \left[e^{+W^2} + \frac{e^{u_s^2} W}{u_s} \right] = 0 \tag{22}$$

$$2We^{-W^2} \left[U(W) + U(u_s) \right] = 1 + \frac{r_s}{r_i} \frac{W}{u_s} \tag{23}$$

If a new parameter, p, be defined by $p \equiv r_s/r_i$, Eq. (23) may then be written as

$$2We^{-W^2} \left[U(W) + U(\log p + W^2) \right] = 1 + \frac{pW}{\sqrt{\log p + W^2}}\ . \tag{24}$$

Tables[4] of $U(x)$ are available. Eq. (24) therefore establishes a useful universal relationship between p, the normalized spot size, and W. It will be noted that W depends only upon the axial beam velocity ($\sim\sqrt{V}$), the initial radial velocity of an outside electron, and the beam current.

Eq. (21) is an equation in the same variables as Eq. (24) but also involves the parameter T, the time of flight of an electron from gun to screen. Parameter T is given by

$$T = \frac{Z}{\sqrt{2\eta V}} \tag{25}$$

Here, Z is the beam length.

The desired relation between V, I, r_i, r_s and Z may be obtained as follows: Define

$$\begin{aligned} f(W,p) &\equiv 2We^{-W^2} \left(U(u_s) + U(W) \right) \\ g(W,p) &\equiv 1 + p\frac{W}{u_s} \end{aligned} \tag{26}$$

4. Terrill, H.M. and L. Sweeney, *J. Frank, Inst.*, Vol. 237, 495 ff, and Vol. 238, 220 ff, 1944.

Select a value of p and plot f and g vs. W. The intersection of f and g occurs at some value of W which may be called β. If the process is repeated for a multitude of values of p, a curve of β vs. p is established.

If, now, W in Eq. (21) be replaced by β, relation (24), the minimization constraint on r_s, is satisfied. Using Eq. (25) to replace T by its equivalent Z yields then

$$\frac{KZ}{\sqrt{2\eta V}} = 2r_i\, e^{-\beta^2}\left[\,U(\beta) + U(u_s(\beta))\right] \qquad (27)$$

In terms of $f(\beta)$ Eq. (27) may also be written

$$\sqrt{\frac{\eta I}{\pi e\sqrt{2\eta V}}}\;\frac{Z}{\sqrt{2\eta V}}\;\frac{1}{r_i} = \frac{f(\beta)}{\beta}\,, \qquad (28)$$

or

$$174\,\frac{I^{1/2}}{V^{3/4}} \cdot \frac{Z}{r_i} = \frac{f(\beta)}{\beta} \qquad (29)$$

Eq. (29) is a universal relationship between V, I, Z, r_i and r_s for the condition that r_s be the smallest possible value. Since β depends only on p ($p = r_s/r_i$), one may plot a single universal curve of

$$\frac{r_s}{r_i} \quad \text{vs.} \quad \frac{I^{1/2}Z}{V^{3/4}r_i}\;.$$

If the focus is adjusted so that the beam has its minimum cross-section, r_o, at the screen a similar universal relationship holds. Both curves are shown in Fig. 3.

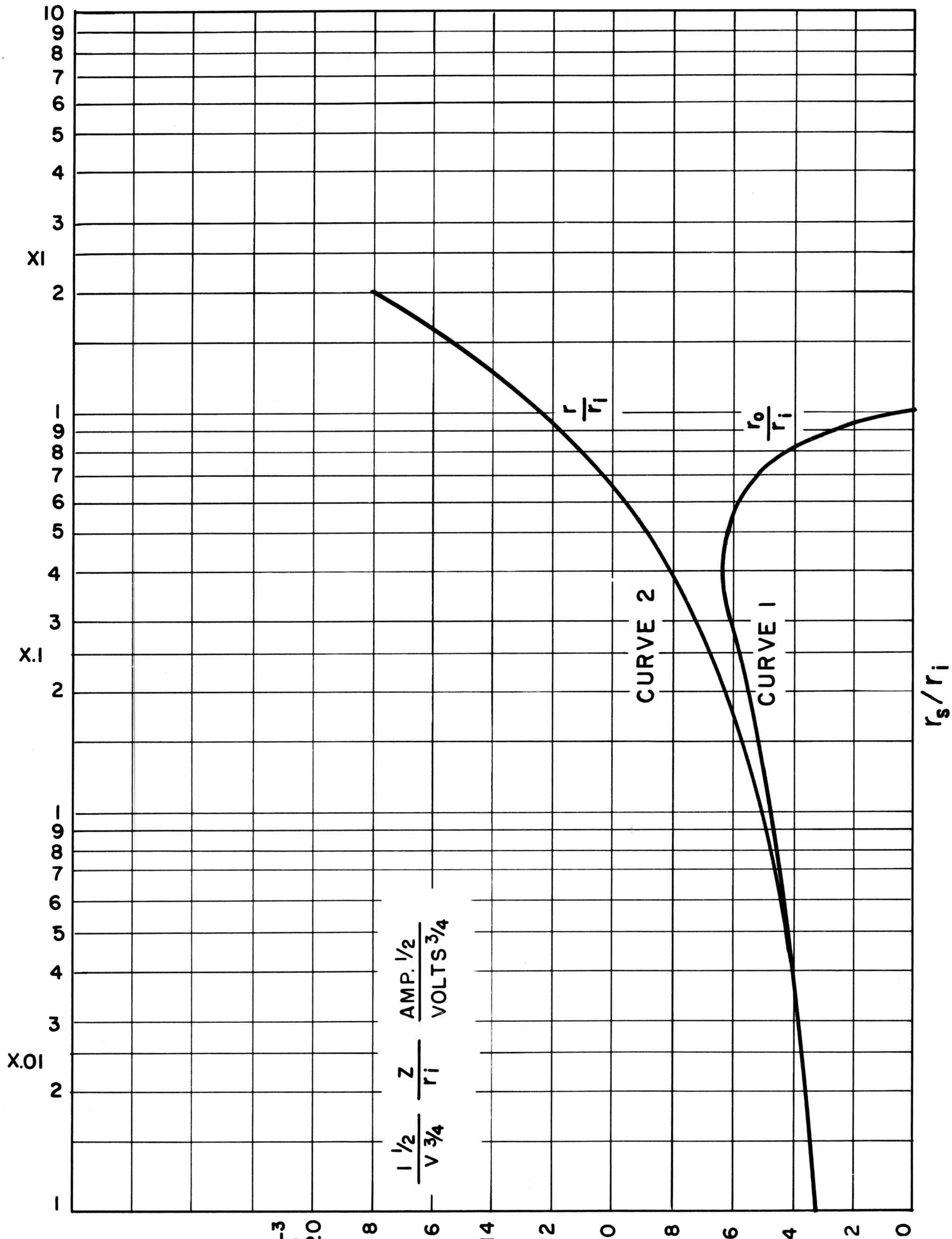

Fig. 2 - Universal relationships for beams with minimum diameter at the screen (curve 1) and for minimum spot size at the screen (curve 2).