J.C. Campbell

GENERAL ELECTRIC

ELECTRONIC COMPONENTS DIVISION

CRT-705-85 (8-57)

TECHNICAL INFORMATION SERIES

Title Page

AUTHOR	SUBJECT CATEGORY	NO. R-58ETC-1
B.A. Findeisen	CATHODE RAY TUBE DEPT.	DATE 3-10-58

TITLE

NUMERICAL METHODS IN CATHODE RAY TUBE ANALYSIS

ABSTRACT

The purpose of this report is to describe certain analytical tools which can be used to good advantage in Cathode Ray Tube Electron Optics problems. These tools consist primarily of numerical methods and computer techniques.

G.E. CLASS	REPRODUCIBLE COPY FILED AT	NO. PAGES
2	Cathode Ray Tube Department Standardizing, Rm. 226, Bld.6 Electronics Park	45
GOV. CLASS		

CONCLUSIONS

The more or less laborious and brute force methods of numerical analysis offer a direct approach to analysing Cathode Ray Tube electron optics problems. Computers and computer techniques offer methods of automatically performing a large portion of the laborious calculations, and make the application of numerical techniques to such problems entirely feasible. Additional methods, such as Kron's method of tearing, allow for rapid accumulation of solutions and should allow the analytical tools to become forerunners to methods of synthesis and design.

By cutting out this rectangle and folding on the center line, the above information can be fitted into a standard card file.

For list of contents—drawings, photos, etc., and for distribution see page attached.

FOR GENERAL TIS INSTRUCTIONS SEE EDI 1.3

GE ACCOUNTING REFERENCE_____

COLLABORATORS_____

COUNTERSIGNED C Dechter DEPT. CRT LOCATION Syracuse, N.Y.

NUMERICAL METHODS IN CATHODE RAY TUBE ANALYSIS

TABLE OF CONTENTS: PAGE

NUMERICAL METHODS IN CATHODE RAY TUBE ANALYSIS

INTRODUCTION:

The purpose of this report is to describe certain analytical tools which can be used to good advantage in Cathode Ray Tube Electron Optics problems. These tools consist primarily of numerical methods and computer techniques, which, of course, apply equally well to all fields of engineering. Some of these methods have been applied to a sample electron lens problem, and the mechanics of the work as well as the results are included in the report. In addition, some logical extensions of the methods are mentioned.

NUMERICAL METHODS:

Numerical methods of solving the differential equations of physics is certainly not a new subject. Such methods and the whole theory of the calculus of finite differences have long been recognized as extremely powerful tools. In fact, such methods offer the only approach to solving complex boundary value problems subject to actual boundaries found in physical devices. Of course, this is the very nature of the Cathode Ray Tube Electron Optics problem, and numerical methods automatically should find a niche in the analysis of these problems. Even though the theory is not new, and the methods have long been recognized as being extremely powerful, there are several reasons why such methods have not been too popular in Engineering analysis up to recent times, to say nothing of Cathode Ray Tube analysis. These reasons are as follows:

1. An enormous amount of dull, monotonous labor is involved in applying numerical techniques to the solution of large practical problems.
2. A so-called "general solution" is not obtained from numerical solutions and hence the results do not conveniently show the effects of variations in problem parameters.

3. A feeling that numerical techniques yield inaccurate approximate results.

The above reasons are probably listed in the order of importance, and if it were not for number 1 the remaining two probably would not exist. Number 1 above will be discussed presently; however, a few words about 2 and 3 are probably in order.

First, a so-called general solution to any problem is very desirable if obtainable. However, if such solutions are not obtainable, certainly numerical solutions cannot be frowned upon. This is particularly true in any field where "cut and try" helped out by analogies is the only practical approach or procedure of today. Then too, it is important to realize that numerical solutions can provide all of the information that classical general solutions provide, if one does not object to the labor involved in cranking through several separate solutions.

Second, the feeling that numerical techniques yield inaccurate approximate results is completely ill-founded. Such techniques actually allow for any desired degree of accuracy, limited only by the amount of labor put forth. Numerical methods likewise allow for the consideration of odd shaped boundaries of finite dimensions, as well as so-called end effects. Therefore, it is very surprising that the very people who approximate physical problems by assuming infinite planes, cylinders etc., and neglect end effects are the ones who condemn the numerical techniques. In addition, the classical methods very rarely allow for a check on the errors involved from such assumpitons; whereas numerical methods constantly allow for error analysis and a direct means of obtaining greater and greater accuracy.

The above statements are not meant as a criticism of classical analysis but rather are made to clear up erroneous impressions which seem to prevail concerning numerical analysis.

In summary it might be stated that there are no real reasons for shying away from numerical analysis other than that given in (1) above.

D I G I T A L _ C O M P U T E R S

USE OF DIGITAL COMPUTERS:

Because of the vast amount of detailed labor which is involved in numerical methods, some means of mechanically performing the necessary operations is certainly desireable. The general purpose digital computers available today are quite readily applicable to this chore. These machines can perform the necessary operations of addition, subtraction, multiplication and division as well as certain logical operations. They can perform these functions with tremendous speed and accuracy and they are finding widespread use in the solution of engineering problems in every field. These machines eliminate much of the undesireable labor when using numerical methods, and are very rapidly becoming the answer to the numerical analysts problems. Presently, however, using or programming these machines becomes quite a task in itself. Nevertheless, the total labor involved is so considerably reduced that the solution of large practical problems has become feasible with the use of present day computers.

Also, new computers and new programming techniques are being developed which continually relieve the programming problem. Hence it appears that the one real reason for not utilizing numerical methods has actually been overcome.

SCOPE OF TREATISE:

Since the field of numerical analysis and the calculus of finite differences is such a broad subject, no attempt will be made to write a complete mathematical treatise. Rather, those interested should refer to standard textbooks (8, 16) on the subject. Likewise, no attempt will be made to apply these techniques to general problems in engineering. Rather, the purpose is to investigate some of these tools and show how they may be applied to electron optics problems. Needless to say, with more time, effort, and experience in this field new and better techniques could be developed.

BOUNDARY VALUE PROBLEMS:

The electrostatic field problem, neglecting any possible space charge, if the first problem of interest in electron optics. It is basically a boundary value problem, where by definition, the potential on the boundaries is known, and it is desired to determine the potential throughout the bounded region. The method of solution by numerical means (which in general is the same in all numerical boundary value problems) can be stated as follows:

1. First write the differential equation to be solved.
2. Replace the derivatives or partial derivatives by their finite difference equivalents. (These can be derived or simply extracted from a table of such equivalents) (Appendix 1)
3. Next set out a number of equally spaced intervals along the various coordinates between the boundaries.
4. Apply the equation found in (2) to each point in the network of points established in (3).
5. Solve the set of algebraic equations relating the spaced functional values to obtain the values of the function at the network points.
6. Finally the value of the function at other points between the mesh points can be found by re-applying the equation obtained in (2).

It should be mentioned that by using unequal intervals near the boundary, a boundary of any arbitrary practical shape can be defined. Where the intervals are unequal the finite difference formulae are different, but the same principle holds. (Appendix 2).

The practical problem of solving the set of algebraic equations as in step (5) is, of course, the one of interest. One method is to actually set up the system of simultaneous equations, one for each point and solve by ordinary techniques. This,

of course, is very laborious. The successful and practical method is to use a system of successive adjustments to initially guessed values. The methods of "relaxation" or "liquidation of residuals" as presented by Southwell (1) work well if the problem is to be solved manually. The method of "iteration", which converges on a solution more slowly, seems to be more appropriate for machine calculations, simply because of the repetitive nature of the process.

If the function being solved for has high order derivatives, several complexities arise and the simple method outlined above must be elaborated. Needless to say, the electrostatic fields encountered in electron topics do have these high order derivatives or "frills" especially in the region near the electrodes.

First, it may be stated that if the order of the finite difference approximation used to replace derivatives in the original equation, is greater than the highest order derivative of the function, then an exact solution is obtained by the above method regardless of spacing or interval size. Therefore, if high order derivatives exist a very elaborate difference equation must be solved. This is inconvenient and impractical. If a low order difference equation is used, extremely close spaced intervals must also be used. (It should be obvious that even a low order difference approximation becomes exact as the interval becomes infinitesimally small). If this method is used it is quite difficult to determine just when the interval spacing is small enough to yield good accuracy. Of course, one sure way, as proposed by Southwell (1) is to halve the interval and obtain another solution. Then if little or no difference is found in the solutions, the desired results have been achieved. This is also an impractical approach, since many solutions may be required before the given criteria are met.

L. Fox (2) first proposed and J.H. Harries (3) later elaborated upon methods of using simple finite difference equations even when high order derivatives

occur in the desired function. These methods involve splitting the solution into partial solutions in one of two ways.

The first method is one of using successively differenced partial solutions. The first solution is differenced to obtain a remainder or error function at each point in the mesh. This error function is then relaxed to obtain a second partial solution etc. The sum of the partial solutions then converges very rapidly on the exact solution. In this modern form relaxation is now regarded as capable of giving any desired degree of accuracy (L. Fox 4), and relaxation in three dimensions actually becomes practical.

The second method involves using a variation of the interval and obtaining several solutions, and then extrapolating for a more exact solution. The accuracy obviously depends upon the degree of the extrapolating polynomial. The real power of the method is realized after proving the symmetry of the situation, and showing that only two solutions are required to make a fifth order approximation (3). In addition to these methods there is wide scope within the framework of numerical analysis for ingenuity and variation in the handling of different problems. Many problems can be attacked by straightforward calculation while others may best be approached by using such artifices as transforms. In any case, once the new variation has been developed, a simple group of straightforward numerical procedures can be set up for engineering computation. These methods in conjunction with present day computing devices offer tremendous tools for analysis.

ELECTROSTATIC FIELDS WITHOUT SPACE CHARGE:

Now turning to the electrostatic field problem itself, and considering a two dimensional case with no space charge, LaPlace's equation must be satisfied.

1. $$\nabla^2 V = 0$$

For the two dimensional case with coordinates x and y 1. becomes:

2. $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Replacing the partial derivatives by corresponding finite difference equivalents with interval spacing h (Appendix 1)

3. $$\frac{V_{-x} + V_{+x} - 2V_o}{h^2} + \frac{V_{-y} + V_{+y} - 2V_o}{h^2} = 0$$

Solving for Vo

4. $$V_o = \frac{1}{4}\left(V_{+x} + V_{-x} + V_{+y} + V_{-y}\right)$$

which is the familiar expression stating that the potential at each point in the field is simply the average of the potential of the four surrounding points. This equation now must be applied to each mesh point in the field which is surrounded by 4 equally spaced mesh points. A different but very similar expression of the form

5. $$V_o = A V_{+x} + B V_{-x} + C V_{-y} + D V_{+y}$$

must be used for points close to the boundaries where non equal intervals are encountered. The method of obtaining the coefficients in equation 5 is shown in Appendix 2.

For the three dimensional case in rectilinear coordinates x, y and z a similar expression to (4) can be derived and is

6. $$V_o = \frac{1}{6}\left(V_{+x} + V_{-x} + V_{+y} + V_{-y} + V_{+z} + V_{-z}\right)$$

For the special case of an axially symmetric field which is extremely useful in Cathode Ray Tube Lens analysis, we start with LaPlace's equation in cylindrical coordinates:

7. $$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

For the axially symmetric case

8. $$\frac{\partial^2 V}{\partial \theta^2} = 0$$

Hence from (7)

9. $$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0$$

Replacing the partial derivatives in (9) by finite difference equivalents

10. $$\frac{V_{-r} + V_{+r} - 2V_0}{h^2} + \frac{h}{r}\frac{(V_{+r} - V_{-r})}{2h^2} + \frac{V_{-z} + V_{+z} - 2V_0}{h^2} = 0$$

Solving for Vo

11. $$V_0 = \frac{1}{4}\left(V_{-r} + V_{+r} + V_{-z} + V_{+z}\right) + \frac{h}{8r}\left(V_{+r} - V_{-r}\right)$$

This equation, of course, applies off the axis of symmetry, whereas on axis when r = 0 the normal three dimensional equation becomes:

12. $$V_0 = \frac{1}{6}\left(V_{-z} + V_{+z} + 4V_r\right)$$

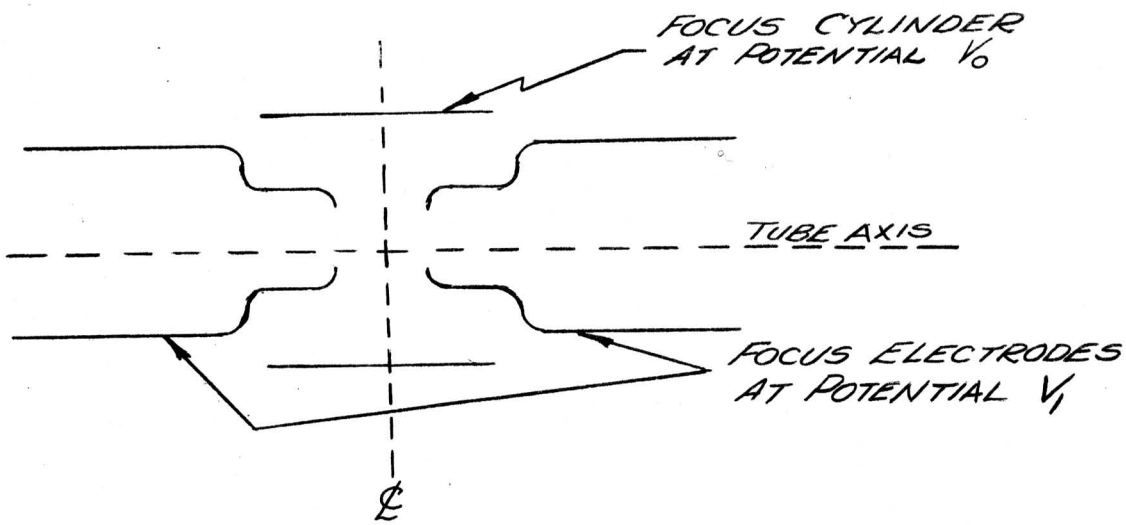
Hence (11) is used off axis while (12) is used on the axis, and the potential function in a plane which includes the axis can be calculated.

If equation (11) is compared to equation (4) the difference between the two dimensional case with no variation in the third dimension can be compared to the axially symmetric three dimensional case. Hence, the right hand term of (11) is seen to be the difference or the error correction term. Looking at this term it is apparent that the correction becomes larger close to the axis, and is also

dependent upon the gradient of the potential perpendicular to the axis. From this it is relatively easy to see that large errors can occur near the axis when using 2 dimensional approximations such as tele deltos paper or rubber membrane analogies. Obviously, this error term must be included point by point and cannot be applied directly to any given 2 dimensional plot. However, it does indicate where errors might be encountered.

SAMPLE PROBLEM:

A sample electron lens configuration was analyzed by the previous methods to become familiar with the mechanics of the problem, as well as the computer techniques which might apply. The lens configuration used is shown below and is similar to a 10UP14A focus lens.



F I G U R E 1.

This lens consists of 3 axially symmetric cylinders as shown. The lens is symmetrical about the lens center line and also about the tube axis; therefore, only one quarter of the electrostatic field need be determined. This one quarter portion of the lens, with two final solutions indicated, is shown in Figure 2.

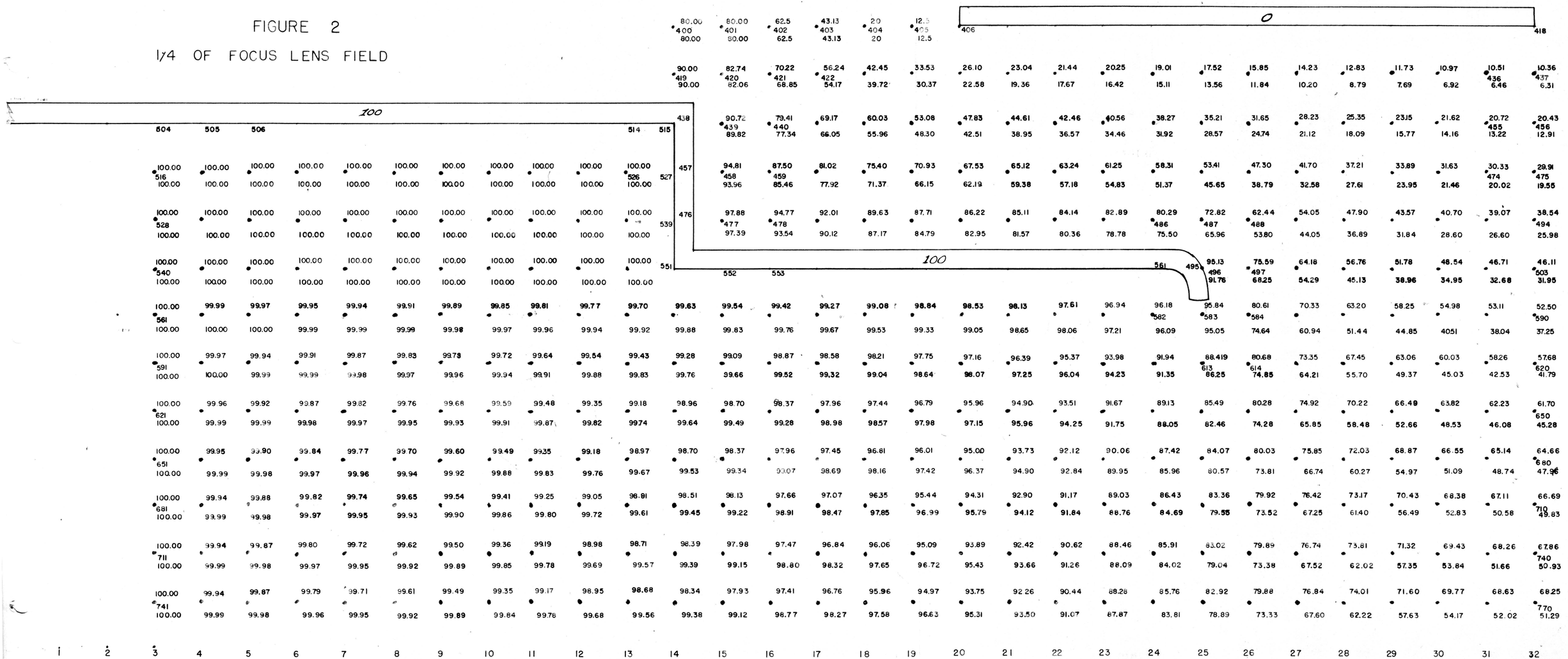
The region of solution is divided into a grid of points with 25 mil spacing. Each point has a numbered position and two values of the potential existing at that point for two different cases. The upper set of numbers indicate the results obtained for a two dimensional case when using equation (4). The lower set of numbers indicate the results obtained for a three dimensional axially symmetric case when using equations (11) and (12).

TWO DIMENSIONAL CASE:

The two dimensional solution was begun with a mesh or grid spacing twice as large as the one shown in Figure 2. The solution was relaxed by hand and was found to be quite inaccurate when differenced as suggested by Harries (3). The closer spaced mesh was then chosen and the problem was programmed for the IBM 650 computer (12). The program was written using the Bell Laboratory Interpretive System (BLIS) (11). Initial values for the potential at the various mesh points were taken from the hand relaxed field plot. The solution was then iterated on the computer 220 times to obtain the solution shown in Figure 2. The iteration process was performed in steps of 10 iterations each, and the potential at several points were plotted vs. the number of iterations to determine when the process should be stopped. The process obviously converged very slowly on a solution. It should be mentioned that the potential is shown in percent with two decimal places; however, the calculations were carried out on the machine with a total of 8 digits or 5 decimal places. At the completion of the iteration process, variations in the 2nd and 3rd decimal place were extremely small for most points in the field, and the solution was judged to be accurate to better than 1% overall for this mesh spacing. By differencing this solution it can be determined that it is not exact and the greatest error occurs in the vicinity of the sharp point of the field electrode. A second partial solution indicates that the region of interest near the tube axis would not be greatly affected, and hence the additional solution was not run to completion.

FIGURE 2

1/4 OF FOCUS LENS FIELD



At this point a few comments about the computer application are in order.

First, the BLIS programming system was chosen because of the relative simplicity of using the method. The BLIS system is an interpretive system whereby rather simple instructions may be given to the machine along with a fixed interpreting routine. These blocks of information are stored in the machine's memory along with the data to be processed. The machine process of calculation then follows the pattern of:

1. Select the appropriate instruction.
2. Interpret the instruction as written and give the machine as many machine language instructions as necessary to accomplish the desired result.
3. Select the next instruction.

Any such interpretive system removes some of the burden from the programmer; however, other disadvantages must be tolerated. The major disadvantages are:

1. A sacrifice of available storage space for the interpretor. (For BLIS 1000 of the 2000 word capacity of the 650 Computer)
2. A sacrifice in calculating speed due to:
    a. Interpreting time each time an instruction is encountered.
    b. The non optimum location of information in the memory.

For the problem under discussion, approximately 250 BLIS type instructions were involved in the program, thereby occupying 250 storage locations. Another 370 locations were used for the 370 points of the grid. Each group of 10 iterations required approximately 20 minutes of computing time. This information is given so that it may be compared with results in later sections.

THREE DIMENSIONAL CASE:

The three dimensional solution was started by assuming initial values for the potential at the various mesh points to be those found in the two dimensional case. The problem was again programmed for the IBM 650 computer; however, this program was written using the Symbolic Optimal Assembly Program (SOAP) (13). The iteration process was carried out in a similar manner to that described for the two dimensional case, and 161 iterations were performed to obtain the solution shown in Figure 2. Again the potential is shown in percent with two decimal places; whereas the calculations were actually performed on the machine with a total of 10 digits or 7 decimal places. The additional two decimal places used in the three dimensional solution are a result of the programming scheme and are not due to any requirement in accuracy. Once again the greatest errors occur in the vicinity of the sharp point of the field electrode; however, no attempt has been made to further refine the solution.

The SOAP programming scheme was chosen for applying the computer to this case due to the added complexity of the calculations and hence a need for more computer storage space. (As mentioned, the BLIS system occupies one half the total available storage locations in the IBM 650 computer). The SOAP program is basically an assembler or compiler type of program whereby a set of symbolic or general instructions are rewritten and assembled into an optimized machine language program. Using the SOAP system involves the following:

1. A group of general instructions are fed into the machine as data.
2. The machine containing the SOAP program computes and operates on the instructions to locate them optimally in storage for a new assembled program.

3. The computer delivers at its output the desired optimally coded program which can be run at any future time.

Any such assembler or compiler system offers the following advantages:

1. The entire storage capacity of the machine is available for the problem at hand since the assembly program is not in the machine at the same time the actual problem is run.
2. Extremely fast operating speeds due to:
    a. Performing only the required calculations. (No interpreting is required since it was essentially performed once and for all in the assemblage process)
    b. The optimum location of information in the memory when the final program is run.

The disadvantage of using SOAP is that the original program must be written in considerable detail. This results in longer programming time, longer programs, and usually more chance for error and hence more program debugging time. Such statements cannot be made about assembler or compiler type programs in general but do apply to SOAP. As machines become larger, more elaborate compilers such as the Formulae Translation (FORTRAN) (15) scheme are being devised to alleviate the programming burden. Such schemes are basically combination interpretors and compilers, and allow a so-called higher type language (less detailed) to be transformed into the lower type (more detailed) language of the machine. These facts are pertinent to the discussion since tremendous progress is being made in this field today, and such techniques make numerical analysis a very appealing tool for the engineer.

For the problem under discussion, approximately 1200 SOAP type instructions were involved in the initial program, and hence approximately 1200 machine language

instructions were involved in the final optimized program. In addition, there were again the 370 mesh point values of the field. Each group of 10 iterations for this case required approximately 8 minutes of computing time. The saving in time due to optimal programming is obvious, especially when it is realized that the three dimensional calculations were much more involved than those in the previous two dimensional case.

CONVERGENCE:

From the two example cases cited, it is seen that a large number of iterations are required before the solution converges upon its final values. In this respect the hand operation of "relaxation" is much more efficient in converging upon a solution. This is true since the slow convergence may be anticipated, and various values may be over corrected to help speed the process. Such techniques, although not impossible, are not economically feasible with present computer facilities and the slower more tedious method of iteration must be relied upon. In general, if a solution is started from initially guessed values the error is large, and then reduces with the number of iterations. A typical curve is shown in Figure 3a.

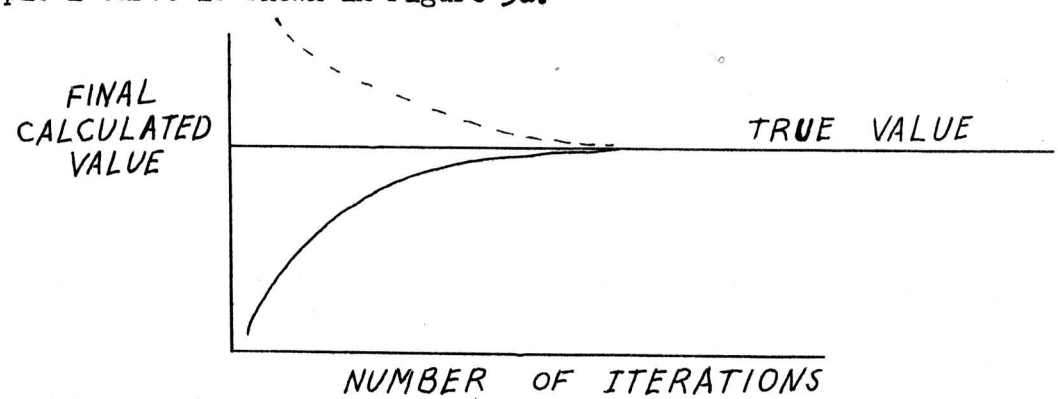


FIGURE 3a

The situation can become worse and a curve as shown in Figure 4a might be encountered.

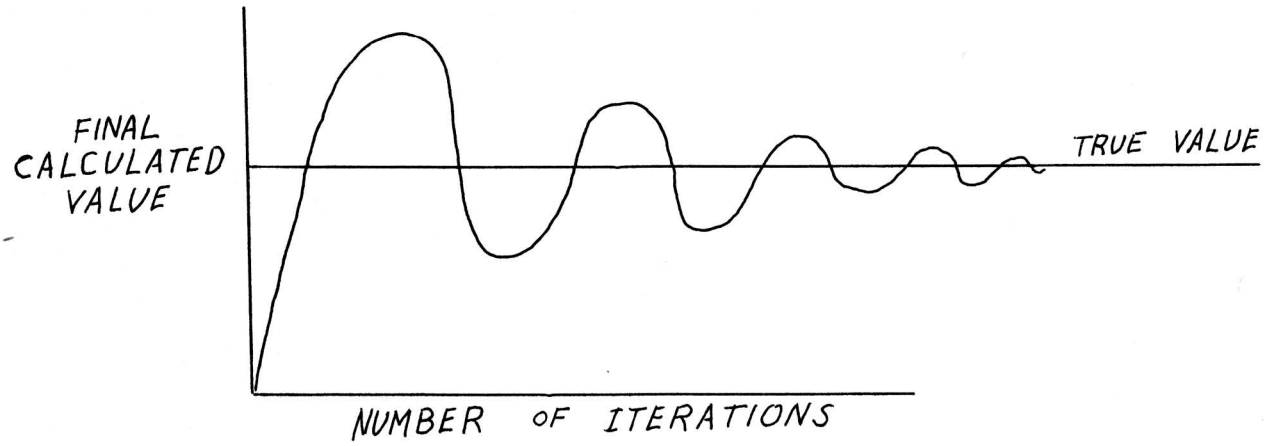


FIGURE 4a

In any event the solution converges asymptotically on the true solution in physically realizable field problems. The point is that a constant check or error analysis must be made to insure proper results.

It is apparent from the asymptotic behavior of the error that the iteration time may be reduced by starting the solution with the best possible initial guess. Good initial values may be obtained by partial relaxation, flux plotting, or from analogies such as the rubber membrane or electrolytic tank etc. Obtaining such initial values may warrant considerable time and effort depending upon the magnitude of the problem; however, refining such answers on the digital computer requires the same programming time, etc. that is necessary to start the solution from zero. Therefore, good judgement must be exercised when comparing the relative cost of computer time versus the set up time for the other methods.

ELECTRON TRAJECTORIES IN ELECTROSTATIC FIELD:

Being able to calculate or plot the electrostatic field for any arbitrary boundary conditions is a large step toward solving many of the electron optics problems. However, this step alone is almost useless and the next major step is to be able to trace electron trajectories through such fields. There are numerous methods of calculating electron paths, some based on the paraxial equation and

others being of a graphical nature. These methods have a restricted application and moreover, there is no convenient way of estimating or controlling the accuracy.

Another method of approach is to integrate the equations of motion, numerically. Again the computer may be used to perform the actual calculations, and once a suitable program has been written, any number of trajectories may be found with ease. Such methods have often been criticized as being too long in application and indeterminate in accuracy; however L.S. Goddard (6) was one of the first to dispel this erroneous notion. Basically the method proposed by Goddard is a step by step method of progressing along the trajectory as is true of all such techniques. However, a considerable portion of the past history of the path is utilized in obtaining the next point or position, and hence the accuracy is increased. In addition, the energy equation can be used as a check and offers a ready means of estimating and controlling the accuracy.

The above method was used in tracing electron paths through the field of the sample problem, and the energy equation check was found to be satisfied over the entire path. A simpler integrating scheme was then used and negligible differences were found between the two methods for the particular field given. Therefore, in the final computer program the simpler integration scheme was used.

It should be mentioned that using either method of calculating trajectories, proper convergence of the rays could not be obtained for the two dimensional field. This discrepancy between calculated results and fact led to the three dimensional field solution where apparently good results have been obtained.

METHOD OF SOLUTION:

Starting with Newton's Law

13. $F = ma$

For the purely electric field

14. $F = -eE$

Considering trajectories in an axial plane equation (14) may be broken into components along the x and y coordinates. Then from (13) and (14)

15. $$m\frac{d^2x}{dt^2} = -eE_x$$

and

16. $$m\frac{d^2y}{dt^2} = -eE_y$$

But since

17. $$\begin{cases} E_x = -\dfrac{\partial V}{\partial x} \\ E_y = -\dfrac{\partial V}{\partial y} \end{cases}$$

equations 15 and 16 become

18. $$m\frac{d^2x}{dt^2} = e\frac{\partial V}{\partial x}$$

19. $$m\frac{d^2y}{dt^2} = e\frac{\partial V}{\partial y}$$

Equations 18 and 19 do not involve the first order derivatives $\frac{dx}{dt}$ OR $\frac{dy}{dt}$) and for purposes of numerical integration several special methods have been devised which take advantage of this fact.

One of these methods, the so-called Fox Noumerov (8) integration formula is

20. $$X_{n+1} = 2X_n - X_{n-1} + \frac{w^2}{12}\left[\ddot{X}_{n-1} + 10\ddot{X}_n + \ddot{X}_{n+1}\right]$$

where $w = t_{i+1} - t_i$ which is the integrating time increment and may be made as small as desired for accuracy.

Of course, a similar expression for the Y direction integration may be written. With the aid of the two equations in the two directions a step by step integration of the path is possible. These are the equations used in the final trajectory program in the sample problem.

In equation 20 it is seen that only two points of the known path are used to determine the next point.

Other formulae due to W.E. Milne (7) and advocated by Goddard (6) are:

$$21.\quad X_{n+1} = X_n + X_{n-2} - X_{n-3} + \frac{w^2}{4}\left(5\ddot{X}_n + 2\ddot{X}_{n-1} + 5\ddot{X}_{n-2}\right)$$

which is exact if fourth differences of x are negligible, and

$$22.\quad X_{n+1} = X_n + X_{n-4} - X_{n-5} + \frac{w^2}{48}\left(67\ddot{X}_n - 8\ddot{X}_{n-1} + 122\ddot{X}_{n-2} - 8\ddot{X}_{n-3} + 67\ddot{X}_{n-4}\right)$$

which is exact if sixth differences of x are negligible.

As stated previously equation 20 is the one actually used, hence it is the only one that will be followed in the report.

From equation 18

$$23.\quad \frac{d^2X}{dt^2} = \ddot{X} = \frac{e}{m}\frac{\partial V}{\partial X}$$

Substituting (23) into (20)

$$24.\quad X_{n+1} = 2X_n - X_{n-1} + \frac{w^2K}{12}\left[\left.\frac{\partial V}{\partial X}\right|_{n-1} + 10\left.\frac{\partial V}{\partial X}\right|_n + \left.\frac{\partial V}{\partial X}\right|_{n+1}\right]$$

Where K is the appropriate constant considering e/m and the units used.

The values in the brackets of equation 24 may, of course, be obtained by differencing the field solution previously calculated. Therefore, by taking any two successive points on an actual, assumed, or otherwise determined path as initial

conditions, and having the various differences of the field solution in both the x and y directions; the desired trajectory can be obtained in a step by step fashion.

Two practical problems are encountered in the method and should be mentioned.

First, the field gradients as obtained by differencing the field solution are only known for the specific mesh points. The equations of motion of the electron are integrated with respect to time, and regardless of the integrating time interval, one cannot be assured that the path points will fall on these mesh positions. This then presents the problem of interpolating for the field gradients at points between the known values of the mesh. Numerous interpolation schemes can be formulated, however, the one adopted for the sample problem is as follows:

A. First the point being considered was found to lie between 4 mesh points where the gradient is known.

B. Next the mesh point closest to the considered point was determined.

C. The mesh point found in (B) was used as a base and straight line interpolation between this point and the two adjacent mesh points were used in the two directions.

In essence a plane is passed through the closest corner and the two adjacent corners of the square mesh, and the gradients at the desired point are determined from this plane.

The second problem can be seen from equation (24). Here it is evident that the point Xn + 1 is being sought, yet the field gradient at this point must be known beforehand. This, of course, is impossible but a good approximation for the field gradient at this point may be obtained by simple extrapolation of the

of the previous x and y increments for the previous time increment. Using this predicted value of field gradient the point Xn + 1 may be calculated. If necessary a better approximation for the field gradient from this point may be substituted into equation 24 and a new calculation made. Such successive approximations, of course, converge on a solution. Fortunately, this repetitious process need not be performed if the increments are small enough.

Perhaps one of the best ways of presenting a clear picture of the logical calculations is by means of an abbreviated block diagram for the flow of information through the computer. Such a diagram is shown in Figure 3.

COMPUTER APPLICATION:

For the problem under consideration an electron trajectory program was written for the IBM 650 computer using SOAP. Approximately 1350 instructions and constants were used in the program, and approximately 45 seconds of computer time are required to trace a complete trajectory through the given lens. Such a trajectory consists of the x and y coordinates of approximately 66 points in the lens region.

The program as written is a more or less universal type program in that electron trajectories can be traced through any known field subject to the following conditions:

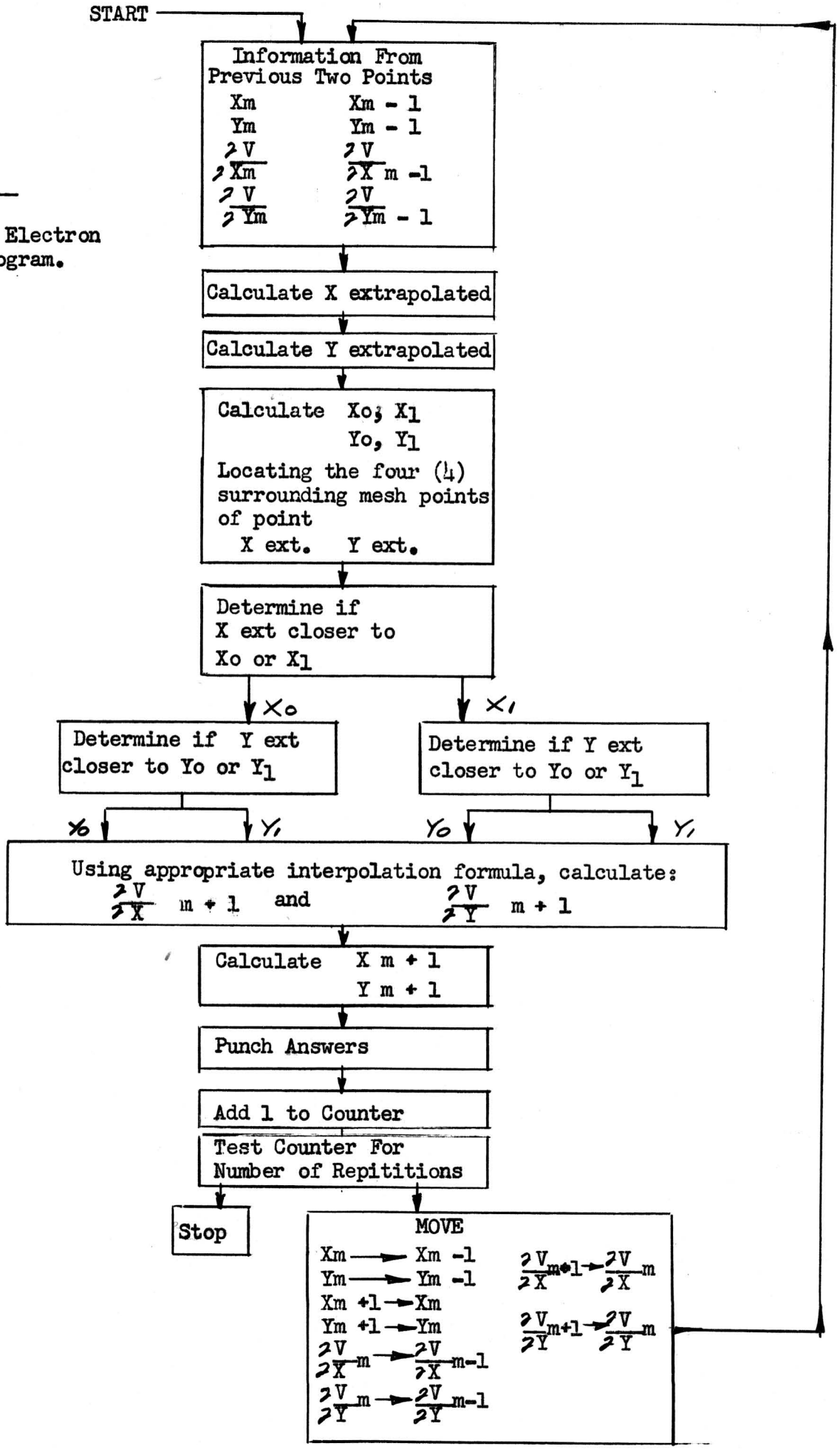
A. The field region and also the electron paths are confined to a square mesh pattern 65 units long and 10 units wide.

B. The field or field gradients are such that the simple interpolation methods used actually apply in the mesh intervals.

C. Obviously the appropriate constants taking care of units, and the appropriate field gradients must be supplied to the program.

FIGURE 3

Flow Chart of Electron Trajectory Program.



It should be emphasized that the above conditions do not severely restrict the use of the program. This is true since any field may be divided by a sufficiently fine mesh to satisfy condition "B". Then condition "A" may be met by dividing the problem into sections. Thus the trajectory may be obtained in one section and provide initial conditions for extending the path into the next section, etc.

Even though other computer programs can be written and there is very little reason for delving into any specific program, it might be wise to explain the reason for the specific mesh pattern handled in this program. A more or less novel coding scheme has been used and it should serve to illustrate some of the many tricks possible when applying digital computers to practical problems.

Some of the pertinent facts of the problem are as follows:

1. An "X" direction gradient and a "Y" direction gradient for each point in the mesh must be available for the calculations.
2. The IBM 650 basic computer has 2000 available storage locations where information may be placed and located. These 2000 locations are addressable by 2000, 4 digit code numbers in the range 0000-1999.
3. The mesh point for which the gradients must be known at any given time is easily calculated from the existing point in the path. However, the gradients themselves must be determined for that mesh point.

   Hence a "Table Look Up Function" is indicated.
4. The path to be calculated extends farther in the x direction than in the y direction.

Because of these facts, the address or position in storage of the various gradients were selected and coded so that they might be located directly from the

coordinates of the mesh points. The first digit was assigned a zero or a one, depending whether the gradient was an "x" or "y" type. The next two digits were allocated to the x coordinate, and the last digit was used for the y coordinate. Therefore, the y coordinate is limited to the 10 positions 0-9 and the x coordinate is unavoidably limited to the 100 positions 00-99. Since the sample problem field did not require 100 increments in the x direction, and since storage space is required for the program instructions themselves, the x values were limited to 65 positions 00-64.

By using this scheme it became a simple matter to generate the required coded address of the desired gradients.

For example the x direction gradient for the mesh point having coordinates

$x = 16$ $y = 3$

would be located in storage position

0163

whereas the y direction gradient for the same point would be located in position 1163

Having formulated the exact locations of the various gradients in this program, it is, of course, quite simple to set up another program for differencing the original field solution. Such a program then yields output cards which act as input cards for the trajectory program. Hence, the entire solution may be performed on the machine. This has been done with the sample problem and basically three programs are involved. The first program iterates the field solution. The second program differences the field solution or calculates the gradients, and also assigns each gradient a coded position for the trajectory calculations. The third program plus the output from the second program then may be used to calculate any desired electron trajectory.

RESULTS OF THE SAMPLE PROBLEM:

Numerous electron trajectories have been calculated for the given lens configuration, and apparently satisfactory focusing action has been obtained. Such trajectory plots in themselves are only of pictorial interest and are not shown in the report. Some of the results of such calculations are of interest and are given in the following.

1. First, parallel rays (parallel to the tube axis) were introduced in the field to determine the apparent focal length of the lens, and also to determine how this focal length varies with lens diameter. In addition, a variation in focus lens voltage was introduced to determine its effects on focal length. A graph showing the results may be seen in Figure 4.

It should be mentioned that the electrons were considered to be in a field free region at the output of the lens field, and straight line extrapolation of the paths was used.

The calculated results show rather marked steps as the path radius is varied. These steps occur at approximately $\frac{1}{2}$ the field mesh spacing and are a result of the interpolation methods used.

One additional item that became evident during the calculations, which is not apparent from Figure 4, is that the curves become flatter as the lens becomes weaker. In other words, the focal length remains more constant with radius for the longer focal length lens. This, of course, agrees with intuitive reasoning.

2. Next, rays emanating from a point source on the tube axis were introduced into the field to observe the focusing action. The problem which immediately arose was where should the fictitious source point be placed. By assumption a point was picked and the results are shown in Figure 5. 10KV was assumed on the field

electrode and 0 volts on the focus cylinder. At a later time experimental measurements on a tube having a somewhat similar focus lens showed that the point source actually should be moved farther back. Another group of trajectories were run using the above voltages and the results are shown in Figure 6.

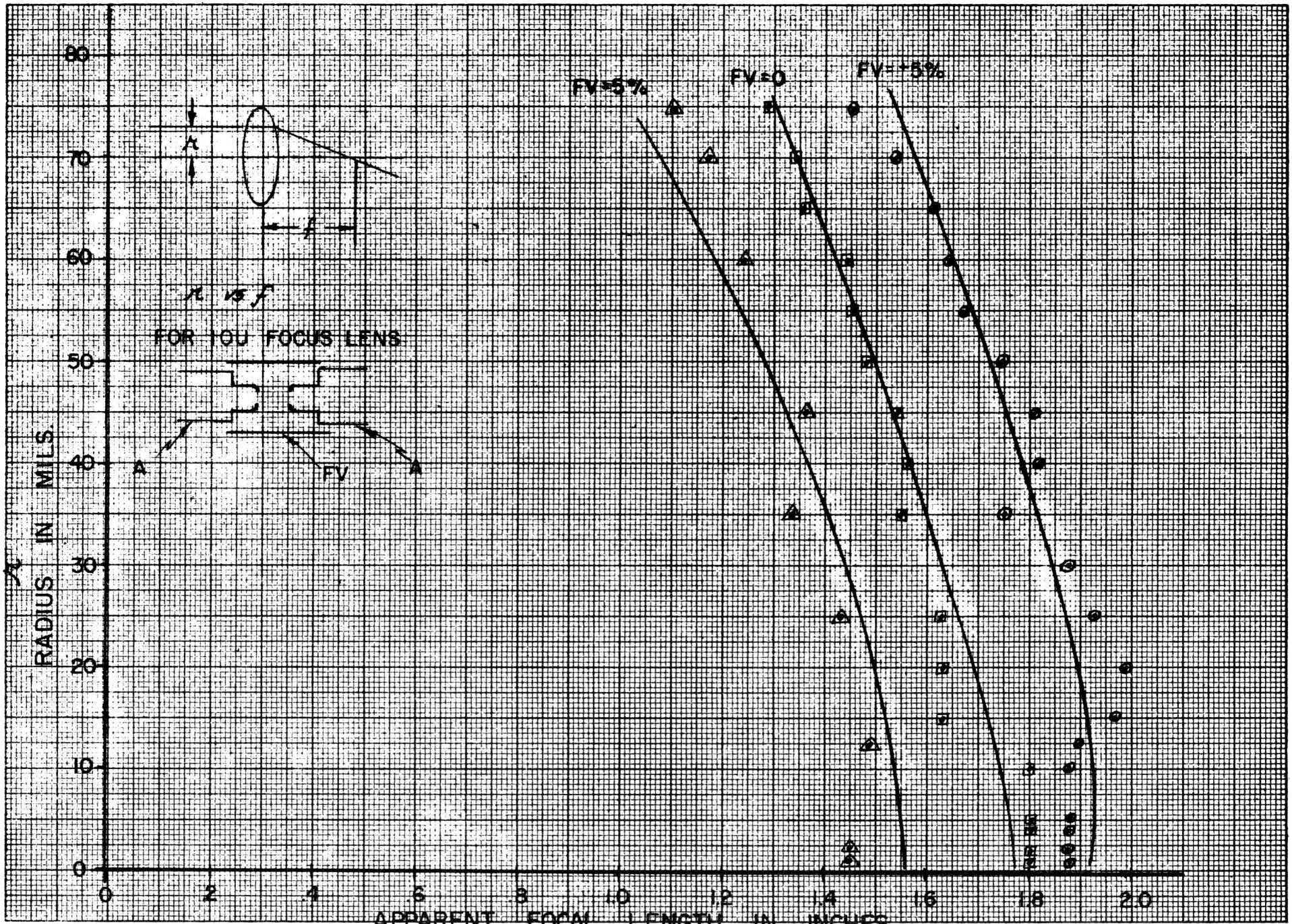
At still a later date, extrapolation of experimental results indicated that the analyzed lens should focus with approximately +1100 volts on the focus cylinder. A group of trajectories were run using 1000 volts on the focus cylinder and the results are shown in Figure 7.

In an attempt to gain even more correlation between the results and experiment, a lens having the same dimensions as the mathematical model is presently under construction.

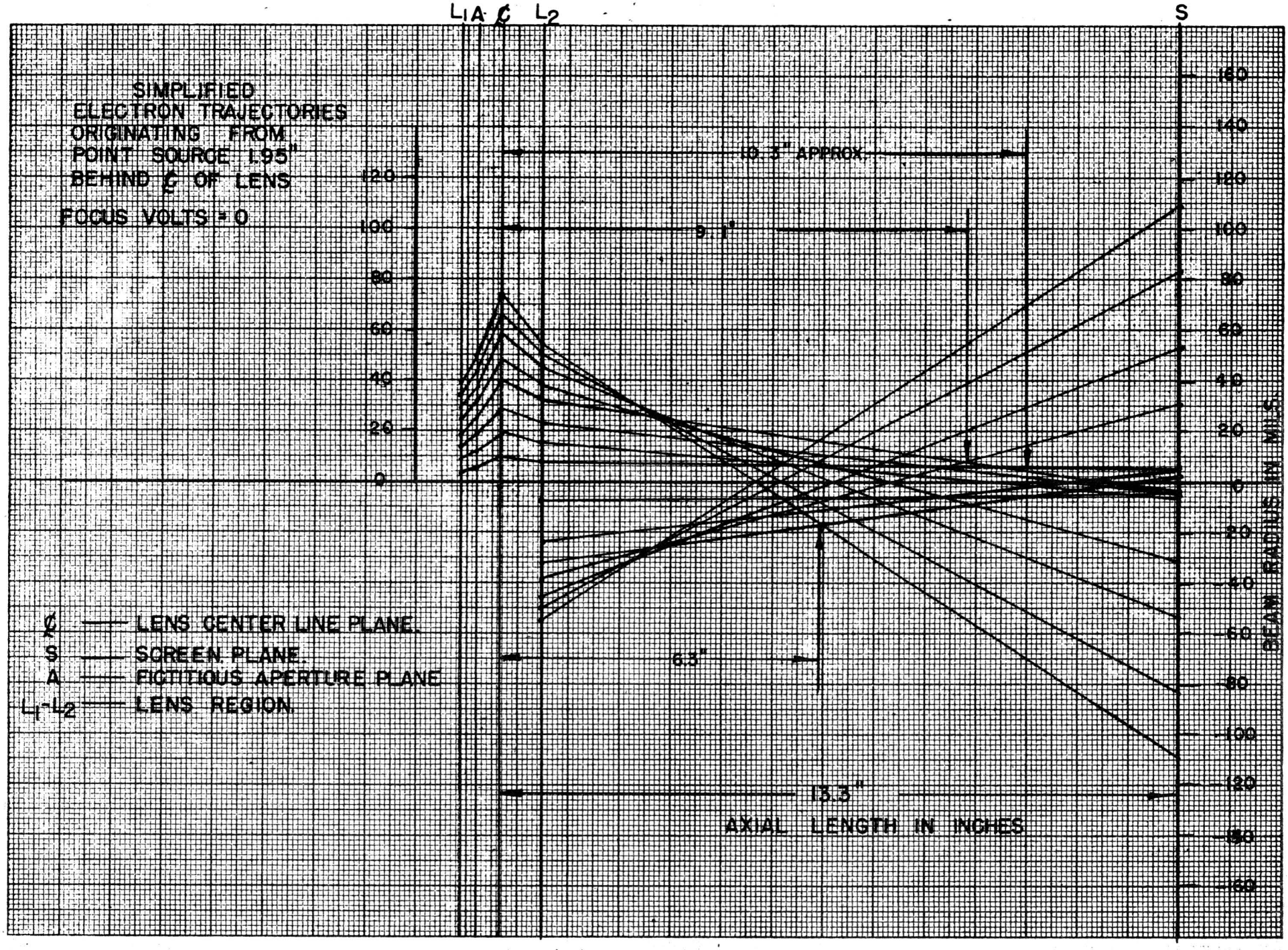
The most that can be said at this time is that the results look extremely encouraging. The analysis was originally undertaken to learn the mechanics of some of the tools and the results are certainly not intended as design data. The results of applying the tools have been extremely enlighteneing, and many more elaborate methods now appear feasible. Furthermore, the application of such techniques appear feasible from a practical, economical standpoint, as well as from the theoretical. Some of the possible extensions of the methods are discussed in the following sections.

ELECTROSTATIC FIELDS WITH SPACE CHARGE:

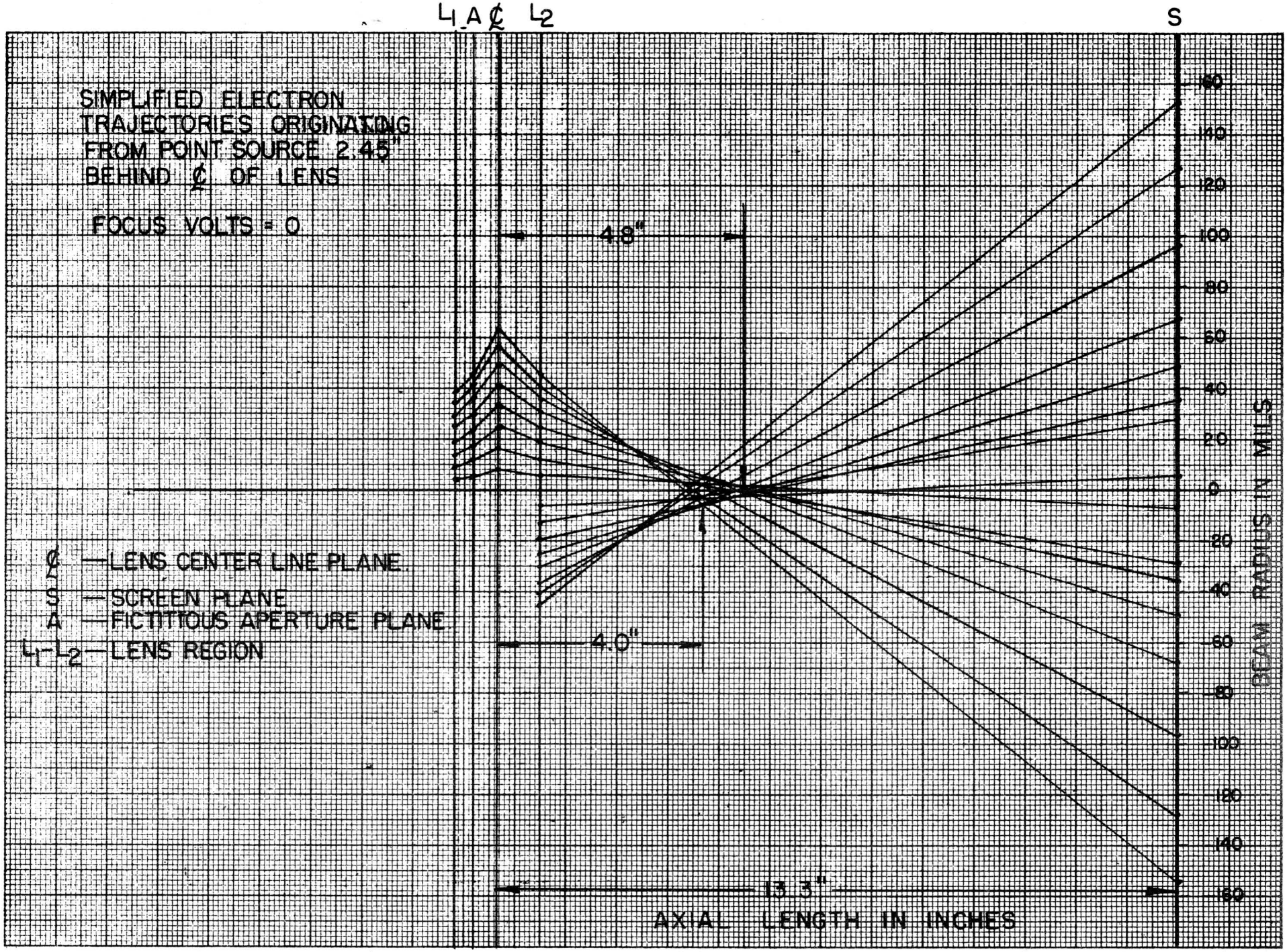
The next problem of interest is the one where space charge is considered. This problem can again be handled by using numerical techniques. The problem becomes a bit more involved; however, once again, after the programming is complete the computations may be carried out automatically. The following outlines the proposed scheme.


FV=5%
FV=0
FV=-5%
A
f
r vs f
FOR 10U FOCUS LENS
A
FV
A
RADIUS IN MILS.
r
0
10
20
30
40
50
60
70
80
0
2
4
6
8
10
12
14
16
18
20
APPARENT FOCAL LENGTH IN INCHES
f

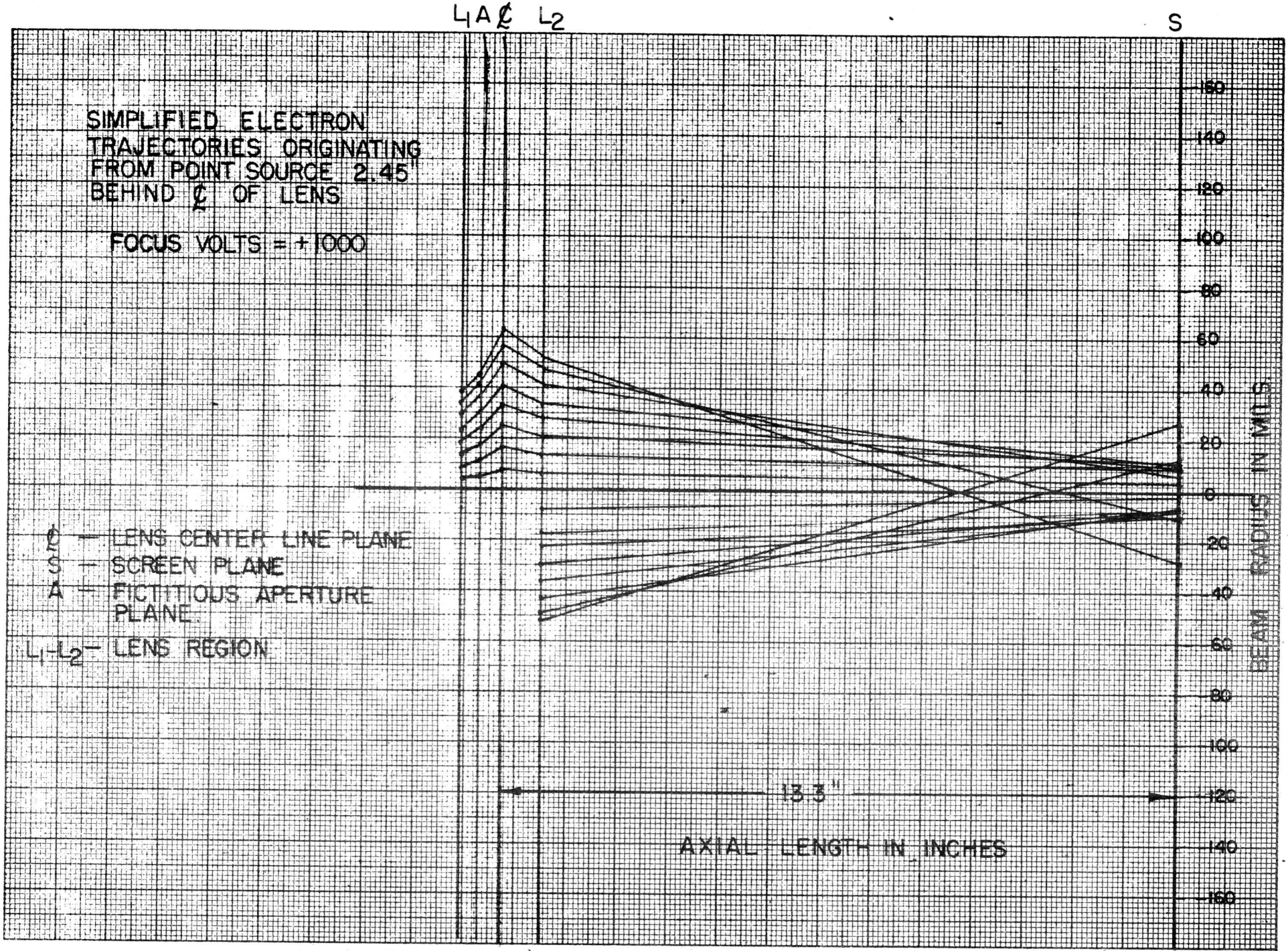
(FIGURE 4)


L1 A ℄ L2
S
SIMPLIFIED
ELECTRON TRAJECTORIES
ORIGINATING FROM
POINT SOURCE 1.95"
BEHIND ℄ OF LENS
FOCUS VOLTS = 0
10.3" APPROX.
9.1"
6.3"
13.3"
AXIAL LENGTH IN INCHES
BEAM RADIUS IN MILS
120
100
80
60
40
20
0
160
140
120
100
80
60
40
20
0
-20
-40
-60
-80
-100
-120
-140
-160
℄ —— LENS CENTER LINE PLANE.
S —— SCREEN PLANE.
A —— FICTITIOUS APERTURE PLANE
L1-L2 —— LENS REGION.

(FIGURE 5)


L1 A ℄ L2
S
SIMPLIFIED ELECTRON TRAJECTORIES ORIGINATING FROM POINT SOURCE 2.45" BEHIND ℄ OF LENS
FOCUS VOLTS = 0
4.8"
4.0"
13.3"
℄ —LENS CENTER LINE PLANE
S —SCREEN PLANE
A —FICTITIOUS APERTURE PLANE
L1-L2—LENS REGION
AXIAL LENGTH IN INCHES
BEAM RADIUS IN MILS
160
140
120
100
80
60
40
20
0
20
40
60
80
100
120
140
160

(FIGURE 6)


L1 A ℄ L2
S
SIMPLIFIED ELECTRON TRAJECTORIES ORIGINATING FROM POINT SOURCE 2.45" BEHIND ℄ OF LENS
FOCUS VOLTS = +1000
℄ — LENS CENTER LINE PLANE
S — SCREEN PLANE
A — FICTITIOUS APERTURE PLANE.
L1-L2 — LENS REGION
13.3"
AXIAL LENGTH IN INCHES
BEAM RADIUS IN MILS
160
140
120
100
80
60
40
20
0
20
40
60
80
100
120
140
160

(FIGURE 7)

When space charge is considered the potential no longer satisfies LaPlace's equation; but must satisfy Poisson's equation

25. $$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

WHERE: $\rho = \frac{\text{COULOMBS}}{\text{METER}^3}$

$\varepsilon = \frac{1}{36\pi} \times 10^{-9} \frac{\text{FARADS}}{\text{METER}}$

Current corresponds to a motion of charges and current density can be written as

26. $$J = \rho v$$

WHERE: $v = \text{METERS/SEC}$

$J = \text{AMPS/METER}^2$

or charge density is

27. $$\rho = \frac{J}{v}$$

but from the energy relation

28. $$\frac{1}{2} m v^2 = eV$$

And the charge velocity v becomes

29. $$v = \sqrt{2 \frac{e}{m} V}$$

Substituting (29) into (27)

30. $$\rho = \frac{J}{\sqrt{2 \frac{e}{m} V}}$$

Substituting (30) into (25)

31. $$\nabla^2 V = -\frac{J}{\varepsilon \sqrt{2 \frac{e}{m}}} V^{-\frac{1}{2}}$$

Now both J and V are in general functions of the coordinates. Hence J must be known throughout the field if equation (31) is to be solved for the potential.

If we now hypothesize the type problem of interest, we can outline a method of solution. The method involves obtaining a series of numerical approximations.

PROBLEM:

Given: (or assumed)

1. Boundaries of field region
2. The input "J" at the boundary

(The problem basically is to bring electrons into an electrostatic field and trace their trajectories through the field, considering both the effects of the boundaries and space charge. Hence, it is obvious that an initial current density of the beam of electrons at the entrance to the field is required.)

METHOD OF SOLUTION:

1. Calculate the space charge free electrostatic field for the given boundary conditions - From LaPlace's equation, or from equation 31 if J = 0
2. From the given initial conditions of the distribution of the electron beam, calculate electron trajectories through the field by integrating the equations of motion.
3. From the calculated electron trajectories calculate the current density J over the entire field region.
4. Using this approximation for "J" recalculate the field from equation (7) Return to Step 2 and repeat steps 2, 3, 4 until no further change in the field or trajectories is encountered (A change will always occur actually - however, the subsequent approximations will converge upon a true solution and the process may be stopped whenever the change is deemed negligible.)

It should be recognized that Poissons equation rather than LaPlace's equation should be programmed from the very start. In this manner the same computer program may be used over and over again on subsequent field solutions. The only additional program required is one which allows for the evaluation of the current density J throughout the region.

MACHINE METHODS OF SOLVING GENERAL FIELD PROBLEM:

As previously stated a more or less general electron trajectory program may be written. Hence, as new and different problems are encountered, only slight modifications in the existing program are required to apply it to the new situation. The same appears to be true of gradient calculating programs and current density evaluating programs. Also it is worth noting that more generalized programs are possible on larger computers with more storage space (IBM 704, etc.)

The major existing problem in applying these numerical methods to general field configurations is the programming of the original boundary value field problem. For each new geometrical configuration involved, a new field calculating program must be written.

To alleviate this programming burden it appears worthwhile to attempt a more sophisticated approach to the general field problem.

GENERAL FIELD PROBLEM:

There are three ways of attacking the generalized field problem from the computer programmers standpoint. They are as follows:

1. Program an assembler or compiler
2. Program an interpreter
3. Program a generalized field problem directly

An assembler or compiler program is one which may be used with a high level language program - - - - and when both are run through the computer a new program in basic machine language is assembled or compiled. The new program can then be run through the computer to solve the given problem. The high level language program must be written for each problem encountered; however, its higher level allows for a simpler less detailed program to be written than is the case in computer language.

An interpreter program is one which may be used with a high level language program - - - and when both are run through the computer a solution is obtained. Hence, an interpreter differs from a compiler in that no new program is written for an additional run through the computer. The interpreter must be used each time a solution is required; therefore, it is much less efficient than a compiler especially if the problem is iterated many times. In other words, the interpreter is in the machine at the same time the problem is run, hence, occupies storage space; and in addition must interpret each high level instruction every time that it is encountered. The compiler interprets the high level instructions once, and compiles an efficient program for later use when the compiler need not be in the machine. Needless to say, the compiler is the more preferable; however, also is the more difficult to program.

The third approach is to program a generalized field problem directly which might be altered for each solution by some initializing procedure. This is an excellent method if the range or class of problems is somewhat restricted. (If boundary shapes, etc. do not change over too wide a range) The more and more generalized the program is to be, the more difficulty is encountered in designing it. In the generalized program the initialization must be written or supplied for each new problem requiring solution.

It is quite difficult to determine which of the foregoing three approaches is most practical. The first method to be studied in more detail will be the direct programming of the generalized field problem. It appears that this approach might lead to more insight to the problem, and also might be the most satisfactory due to the restricted class of problems involved in the Cathode Ray Tube Department.

KRONS METHOD (DIAKOPTICS)

The type of problems being considered here can, of course, become immense in magnitude, and it is not too difficult to visualize problems of such a size as to be several steps ahead of present day computers. This brings us to another powerful tool developed by Gabriel Kron. The tool or method called Diakoptics is essentially one of tearing large complex and interconnected systems into simpler pieces for analysis. These smaller parts can be solved on existing computers. The solution for the complete problem may then be obtained by interconnecting the solutions of the smaller subdivisions. An added advantage is realizable since if alterations are made in any of the sub-parts, only these parts are recalculated and then interconnected with the existing solutions for the remaining parts.

The technique of tearing leads to a system of filing solutions as they are acquired. Then on some future occasion, several of the solved systems on file can be interconnected to form solutions for new systems. In this way the library of solutions pyramids and becomes more and more useful. Also, unnecessary time need not be spent in resolving previously solved portions of new problems.

For the Cathode Ray Tube problems of interest, solutions for straight cylinders may be obtained as well as for various flares and terminating curls. Then these solutions may be combined in any manner, at will, to obtain solutions for new configurations.

Kron's method is truly a powerful engineering tool; however, it should not be believed that as soon as the question of piecewise solution of problems is raised, one can proceed and immediately accomplish the task. The method involves much laborious work and certainly would entail many soltuions similar to the one described in this report. However, the simple fact remains, the method is available.

Kron's method utilizes the theory of tensors, topological models in the form of electrical circuits, and the tearing apart of such circuits physically or functionally. The use of topological models or equivalent circuits as opposed to the direct use of difference equations allows for the physical tearing of the problem, and subsequently to the formulation of an interconnection scheme. The mechanics of solving the various sub-divisions of the problem are similar to those involved in this report.

These methods have not been completely explored, and definitely a considerable amount of "learning time" would be involved in attempting to apply the techniques to Cathode Ray Tube Problems. Nevertheless, this tool of Diakoptics offers tremendous possibilities.

CONCLUSIONS:

From most of the foregoing it should be evident that the electron optics problems associated with Cathode Ray Tubes are quite complex. The more or less laborious and brute force methods of numerical analysis offer a direct approach to analyzing these problems. Computers and computer techniques offer methods of automatically performing a large portion of the laborious calculations, and make the application of numerical techniques to such problems entirely feasible. Additional methods, such as Kron's method of tearing, allow for rapid accumulation of

solutions and should allow the analytical tools to become forerunners to methods of synthesis and design.

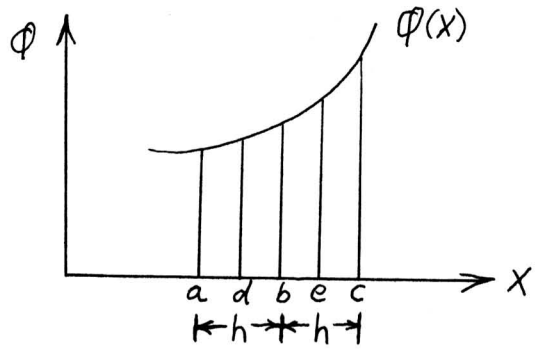
BAF/rad

APPENDIX I

Finite Difference Equivalents:

For the uninitiated, a brief description of finite difference equations and their derivation is included in this appendix. This material was extracted from a report by J. H. Owen Harries (3) and is originally due to Shaw (5).

Setting up difference equations:



In Figure 1 let the curve shown by the graph of some function. $\phi = \phi(x)$ At some point x = xe, we have be definition for the point e of Figure 1.

FIGURE 1

1. $$\left(\frac{d\phi}{dx}\right)_e = \lim_{h\to 0} \frac{\phi(x_e+\frac{h}{2}) - \phi(x_e-\frac{h}{2})}{h}$$

$$= \lim_{h\to 0} \frac{\phi_c - \phi_b}{h}$$

where by $(d\phi/dx)_e$ is meant the value of $\frac{d\phi}{dx}$ at the point x=e etc.

Consequently, when h is small, we may write for equation 1:

2. $$\left(\frac{d\phi}{dx}\right)_e \approx \frac{\phi_c - \phi_b}{h}$$

and similarly

3. $$\left(\frac{d\phi}{dx}\right)_d \approx \frac{\phi_b - \phi_a}{h}$$

Also since

4. $$\frac{d^2\phi}{dx^2} = \frac{d}{dx}\left(\frac{d\phi}{dx}\right)$$

it follows that

5. $$\left(\frac{d^2\varphi}{dx^2}\right)_b \approx \frac{\left(\frac{d\varphi}{dx}\right)_e - \left(\frac{d\varphi}{dx}\right)_d}{h}$$

By introducing values from equations 2 and 3 one obtains:

6. $$\left(\frac{d^2\varphi}{dx^2}\right)_b \approx \frac{\varphi_c + \varphi_a - 2\varphi_b}{h^2}$$

In equation 6. the expression $\frac{\varphi_c + \varphi_a - 2\varphi_b}{h^2}$ is the finite difference approximation for: $\left(\frac{d^2\varphi}{dx^2}\right)_b$

Check by Taylor's Series:

Alternatively, on the assumption that locally the function φ may, be expanded in the form of a Taylor's series, we may write:

7. $$(\varphi)_x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \text{- - - - -}$$

with x = xb as origin we then obtain

$$a_0 = \varphi_b$$

$$a_1 = \left(\frac{d\varphi}{dx}\right)_b$$

$$2!\, a_2 = \left(\frac{d^2\varphi}{dx^2}\right)_b$$

$$3!\, a_3 = \left(\frac{d^3\varphi}{dx^3}\right)_b$$

and substituting $(x_c - x_b) = h$, $(x_a - x_b) = -h$ etc. and solving for a_2 we obtain

8. $$2!\, a_2 = \left(\frac{d^2\varphi}{dx^2}\right)_b = \frac{\varphi_c + \varphi_a - 2\varphi_b}{h^2} + \frac{h^2}{12}\left(\frac{d^4\varphi}{dx^4}\right)_b + \text{- - - - - -}$$

so that the finite difference approximation for $d^2\varphi/dx^2$ as given by equation 6 neglects the terms $\frac{h^2}{12}\left(\frac{d^4\varphi}{dx^4}\right) + \text{- - - - -}$

The Taylor series expansion method may always be used to determine finite difference approximations and the errors involved in such approximations.

Difference Equivalents:

Consider now the function $\phi = \phi(x,y)$ as a function of two independent variables. If, on the x, y plane, a mesh of lines be drawn parallel to the x and y axes, and spaced a distance h apart, as in Figure 2, then in the manner just outlined the following approximations are obtained.

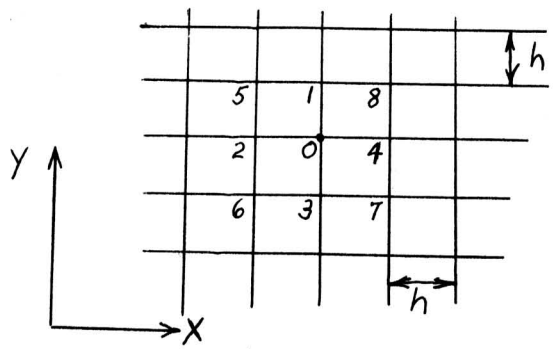


FIGURE 2

when points 2 and 4 are 2h apart

$$\left(\frac{\partial \phi}{\partial x}\right)_0 \approx \frac{\phi_4 - \phi_2}{2h}$$

9.

$$\left(\frac{\partial \phi}{\partial y}\right)_0 \approx \frac{\phi_1 - \phi_3}{2h}$$

From equation 6 the 2 dimensional LaPlacian is

$$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)_0 \approx \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0}{h^2}$$

In a similar manner any differential equation or partial differential equation may be approximated by finite difference equivalents.

A P P E N D I X II

Unequal intervals in finite difference equations:

The material in Appendix II has been extracted from a report by J. H. Owen Harries (3) and is originally due to Shaw (5).

Curved Boundaries:

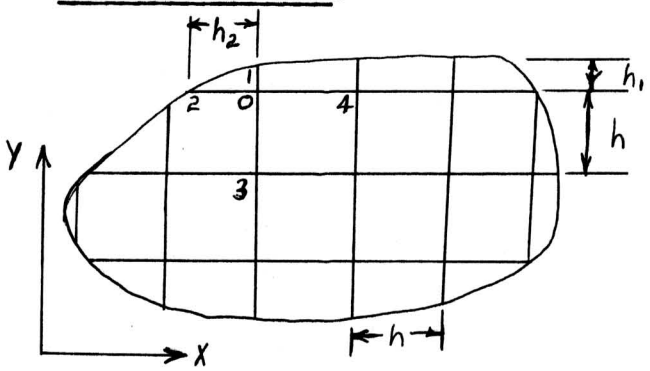


FIGURE I

In many of the boundary value problems encountered in practice the boundaries are curved, so that adjacent to the boundary the treament as given in Appendix I is incorrect. Let Figure I represent one such problem.

Correcting the Equations:

On covering the region with a square mesh, points like that marked 0 occur, that is, points in which one or more of the associated arms 01, 02, 03, or 04, is less than the standard length h. For such points it becomes necessary to develop special equations.

Consider the polynomial

1. $$\phi = \phi_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy$$

Thus, at the point 0 (x = 0, y = 0) we have

2. $$\left(\frac{\partial^2 \phi}{\partial x^2}\right)_0 = 2a_3 \quad ; \quad \left(\frac{\partial^2 \phi}{\partial y^2}\right)_0 = 2a_4$$

Then referring to Figure 1, substituting the irregular arms h1 and h2 and solving for a_3 and a_4 we find that

3. $$\left(\frac{\partial^2 \phi}{\partial x^2}\right)_0 = 2\,\frac{h(\phi_1 - \phi_0) + h_1(\phi_3 - \phi_0)}{h h_1 (h + h_1)}$$

and

4. $$\left(\frac{\partial^2\phi}{\partial y^2}\right)_0 = 2\,\frac{h(\phi_2-\phi_0)+h_2(\phi_4-\phi_0)}{hh_2(h+h_2)}$$

So that $(\nabla^2\phi)_0$ may be written as

5. $$h^2(\nabla^2\phi)_0 = \frac{2}{\alpha_1(1+\alpha_1)}\phi_1 + \frac{2}{(1+\alpha_1)}\phi_3 + \frac{2}{\alpha_2(1+\alpha_2)}\phi_2 + \frac{2}{(1+\alpha_2)}\phi_4 - \left(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}\right)\phi_0$$

where $\alpha_1 = h_1/h$, $\alpha_2 = h_2/h$ Hence for $0 < \alpha < 1$

We have:

6. $$h^2(\nabla^2\phi)_0 = A\phi_1 + B\phi_2 + C\phi_3 + D\phi_4 - (E+F)\phi_0$$

Where the values for A - - - F may be calculated and tabulated once and for all. Such a table may be found in Reference 5.

B I B L I O G R A P H Y

1. R.V. Southwell, "Relaxation Methods in Engineering Science" (Clarendon Press, Oxford, 1940); also"Relaxation Methods in Theoretical Physics (1946

2. L. Fox, Proc. Royal Soc. Vol. 190 (1947) pp. 31-59

3. J. H. Owen Harries, Unpublished report submitted to Cathode Ray Tube Department General Electric Co.(Progress Report No. 40A dated 8/15/52)

4. L. Fox, Proc. Cambridge Phil. Soc. Vol 45., Part I, pp. 50-68

5. F. S. Shaw, "Relaxation Methods (Dover Publications, Inc. 1953); also "The Torsion of Solid and Hollow Prisms". Report ACA-11, Australian Council of Aeronautics. November 1944.

6. L.S. Goddard, Proc. Phys. Soc. Vol. 56 (1944) pp. 372-378

7. W.E. Milne, 1933 Amer. Math. Monthly 40, 322

8. Salvadori & Baron "Numerical Methods in Engineering" (Prentice Hall 1952)

9. G. Kron, "Tensor Analysis of Networks" (John Wiley & Sons, New York)

10. M.S. Pendergast Jr., "Do It Yourself Diakoptics" unnumbered General Electric report obtainable from GE Library, Tech. Data Center, Schenectady, NY

11. IBM Technical Newsletter No. 11 (1956)

12. IBM Manual, 650 Magnetic Drum Data Processing Machine (1955)

13. IBM Manual, SOAP II (1957)

14. IBM Manual, IBM Electronic Data Processing Machines Type 704 (1956)

15. IBM Manual, FORTRAN (1956)

16. J. B Scarborough, "Numerical Mathematical Analysis" (The Johns Hopkins Press 1955)